Dynamical Casimir Effect and the Black Body Spectrum

Jaume Haro[a,1] and Emilio Elizalde[b,2]

[a]*Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya*
Diagonal 647, 08028 Barcelona, Spain
[b]*Instituto de Ciencias del Espacio (CSIC) and Institut d'Estudis Espacials de Catalunya (IEEC/CSIC)*
Universitat Autònoma de Barcelona, Torre C5-Parell-2a planta, 08193 Bellaterra (Barcelona) Spain

To Klaus Fredenhagen on the occasion of his 60th Birthday: Herzlicher Glückwunsch!

Abstract

Creation of scalar massless particles in two-dimensional Minkowski space-time—as predicted by the dynamical Casimir effect—is studied for the case of a semitransparent mirror initially at rest, then accelerating for some finite time, along a specified trajectory, and finally moving with constant velocity. When the reflection and transmission coefficients are those in the model proposed by Barton, Calogeracos, and Nicolaevici [$r(w) = -i\alpha/(\omega + i\alpha)$ and $s(w) = \omega/(\omega+i\alpha)$, with $\alpha \geq 0$], the Bogoliubov coefficients on the back side of the mirror can be computed exactly. This allows us to prove that, when α is very large (case of an ideal, perfectly reflecting mirror) a thermal emission of scalar massless particles obeying Bose-Einstein statistics is radiated from the mirror (a black body radiation), in accordance with previous results in the literature. However, when α is finite (semitransparent mirror, a physically realistic situation) the striking result is obtained that the thermal emission of scalar massless particles obeys Fermi-Dirac statistics. Possible consequences of this result are envisaged.

PACS: 03.70.+k, 04.62.+v, 42.50.Lc, 11.10.Ef

1. Introduction. The Davies-Fulling model [1, 2] describes the creation of scalar massless particles by a moving perfect mirror following a prescribed trajectory. This phenomenon is also termed as the dynamical Casimir effect. Recently, the authors of the present paper introduced a Hamiltonian formulation in order to address some problems associated with the physical description of this effect at any time while the mirror is moving [3]; in particular, of the regularization procedure, which turns out to be decisive for the correct derivation of physically meaningful quantities. A basic difference with previous results was that the motion force derived within the new approach contains a reactive term—proportional to the mirror's acceleration. This term is of the essence in order to obtain particles with a positive energy all the time while the oscillation of the mirror takes place, and always satisfying the energy conservation law. Such result followed essentially from the introduction of physically realistic conditions, e.g. a partially transmitting mirror, which becomes transparent to very high frequencies.

Here we will study a different aspect of the introduction of physical, semitransparent mirrors, namely the particle spectrum produced—in the conditions of the Fulling-Davies effect—by a mirror of this sort which is initially at rest, then accelerates during a large enough (but finite) time span, u_0, along the trajectory defined in [4, 5] (known to lead to a Planck spectrum):

$$v = \frac{1}{k}(1 - e^{-ku}) \tag{1}$$

[1]E-mail: jaime.haro@upc.edu
[2]E-mail: elizalde@ieec.uab.es, elizalde@math.mit.edu

(in light-like coordinates, where k is some frequency), and finally, for $u \geq u_0$, is left alone moving with constant velocity in an inertial trajectory.

Our interest will be to calculate the radiation emitted by the mirror from its back (right) side. As is well-known, a perfect mirror that follows this kind of trajectory produces a thermal emission of scalar massless particles obeying Bose-Einstein statistics. More precisely, for $1 \ll \omega'/k \ll e^{ku_0}$ and $1 \ll \omega'/\omega \ll e^{ku_0}$ (with ω' the frequency of an ingoing and ω of an outgoing particle, respectively), the square of the β-Bogoliubov coefficient satisfies [6, 7, 8]

$$\left|\beta^{R,R}_{\omega,\omega'}\right|^2 \equiv \left|(\phi^{out\,*}_{\omega,R};\phi^{in}_{\omega',R})\right|^2 \cong \frac{1}{2\pi\omega' k}\left(e^{2\pi\omega/k}-1\right)^{-1}, \tag{2}$$

where this square of the β-Bogoliubov coefficient gives the average number of produced particles in the ω mode per unit of frequency. That is, the average number of produced particles in the ω mode, denoted $\mathcal{N}_\omega$, is given by $\mathcal{N}_\omega = \int_0^\infty d\omega' \left|\beta^{R,R}_{\omega,\omega'}\right|^2$.

Here ω' denotes the frequency of an ingoing particle (a particle coming from the past infinity), and ω the frequency of the outgoing one (particle going to the future infinity). Note that for the trajectory (1) the ingoing mode suffers, after the scattering, a very high redshift, for this reason, in order to obtain (2), we need the above conditions (see for details [8], and Section (3a)).

Turning to the case of a partially reflecting mirror—in which we will be mainly interested in this paper—in order to obtain the radiation on the right hand side (rhs) of the mirror, we also need to calculate the corresponding Bogoliubov coefficient, in this case: $\beta^{R,L}_{\omega,\omega'} \equiv (\phi^{out\,*}_{\omega,R};\phi^{in}_{\omega',L})^*$.

We thus first obtain the 'in' modes on the rhs of the mirror when the reflection and transmission coefficients are $r(w) = \frac{-i\alpha}{\omega+i\alpha}$ and $s(w) = \frac{\omega}{\omega+i\alpha}$, with $\alpha \geq 0$, that is, when the Lagrangian density is given by [9, 10, 11]

$$\mathcal{L} = \frac{1}{2}[(\partial_t\phi)^2 - (\partial_z\phi)^2] - \alpha\sqrt{1-\dot{g}^2(t)}\phi^2\delta(z-g(t)), \tag{3}$$

where $z = g(t)$ is the trajectory in the (t,z) coordinates.

2. Main results. The main results of this paper, some of them quite remarkable, are the following (for $1 \ll \omega'/k \ll e^{ku_0}$ and $1 \ll \omega'/\omega \ll e^{ku_0}$).

1. In the perfectly reflecting case, i.e., when $\omega' \ll \alpha$, we obtain

$$\left|\beta^{R,R}_{\omega,\omega'}\right|^2 \cong \frac{1}{2\pi\omega' k}\left(e^{2\pi\omega/k}-1\right)^{-1}, \quad \left|\beta^{R,L}_{\omega,\omega'}\right|^2 \cong 0, \tag{4}$$

that is, a thermal radiation of massless particles obeying Bose-Einstein statistics arises.

2. In the perfectly transparent case, i.e.,when $\alpha \cong 0$, we have

$$|\beta^{R,R}_{\omega,\omega'}|^2 \cong 0, \quad |\beta^{R,L}_{\omega,\omega'}|^2 \cong 0. \tag{5}$$

In other words, there is no particle production.

3. In the physically more realistic case of a partially transmitting mirror (transparent to high enough frequencies [3]), i.e., when $\alpha \ll \omega'$, what we obtain is

$$\begin{aligned}\left|\beta^{R,R}_{\omega,\omega'}\right|^2 &\cong \frac{1}{2\pi\omega k}\left(\frac{\alpha}{\omega'}\right)^2\left(e^{2\pi\omega/k}+1\right)^{-1},\\ \left|\beta^{R,L}_{\omega,\omega'}\right|^2 &\sim \frac{1}{\omega\omega'}\mathcal{O}\left[\left(\frac{\alpha}{\omega'}\right)^2\right].\end{aligned} \tag{6}$$

And, since $\left|\beta^{R,L}_{\omega,\omega'}\right| \ll \left|\beta^{R,R}_{\omega,\omega'}\right|$, we conclude quite surprisingly that a semitransparent mirror emits a thermal radiation of scalar massless particles obeying Fermi-Dirac statistics.

Here it is important to emphasize that the word 'statistics' refers to the β-Bogoliubov coefficient characterizing the spectrum of the radiated particles and not to the algebra obeyed by the creation and annihilation operators, that always satisfy the canonical anti-commutation relations. That is, the original particles are bosons, but the spectrum of the radiated emission corresponds to fermionic ones. This could have some bearing on the local algebraic description of quantum fields [12].

Given the novelty and potential importance of this result, we thought we should devote the rest of the paper to provide a rigorous and systematic proof of the same. Also, we will give hints to possible interesting consequences and applications of our finding.

3. Proof of the results. (3a) Perfectly reflecting, moving mirror. Consider a massless scalar field ϕ in 2-dimensional Minkowski space-time. Assume that the mirror trajectory is $\mathcal{C}^1$ (once continuously differentiable), and that it has the following form, in the light-like coordinates $u \equiv t - z$ and $v \equiv t + z$,

$$v = V(u) \equiv \begin{cases} u, & \text{if} \quad u \leq 0, \\ \frac{1}{k}(1 - e^{-ku}), & \text{if} \quad 0 \leq u \leq u_0, \\ V(u_0) + A(u - u_0), & \text{if} \quad u \geq u_0, \end{cases} \tag{7}$$

with $A = e^{-ku_0}$. We also assume that $u_0 \gg 1$. Note that this trajectory can be written under the following form, too

$$u = U(v) \equiv \begin{cases} v, & \text{if} \quad v \leq 0, \\ -\frac{1}{k}\ln(1 - kv), & \text{if} \quad 0 \leq v \leq v_0, \\ U(v_0) + A^{-1}(v - v_0), & \text{if} \quad v \geq v_0. \end{cases} \tag{8}$$

For a perfectly reflecting mirror, the set of 'in' and 'out' mode functions on the rhs of the mirror is [13]

$$\begin{aligned} \phi^{in}_{\omega,R}(u,v) &= \tfrac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega v} - e^{-i\omega V(u)}\right)\theta(v - V(u)), \\ \phi^{out}_{\omega,L}(u,v) &= \tfrac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega u} - e^{-i\omega U(v)}\right)\theta(v - V(u)). \end{aligned} \tag{9}$$

Our main aim now is to calculate the Bogoliubov beta coefficient

$$\beta^{R,R}_{\omega,\omega'} \equiv \left(\phi^{out\,*}_{\omega,R}; \phi^{in}_{\omega',R}\right)^*, \quad \omega, \omega' > 0, \tag{10}$$

where the parenthesis on the rhs denotes the usual product for scalar fields [14].

In order to compute this coefficient we choose the right null future infinity domain $\mathcal{J}^+_R$; since the trajectory is $\mathcal{C}^1$, we have

$$\begin{aligned} \beta^{R,R}_{\omega,\omega'} &= 2i\int_{\mathbb{R}} du\, \phi^{out}_{\omega,R}\partial_u\phi^{in}_{\omega',R} = \frac{1}{2\pi i\sqrt{\omega\omega'}}\frac{\omega'}{\omega + \omega'} \\ &\quad - \frac{1}{2\pi i\sqrt{\omega\omega'}} e^{-i\omega u_0} e^{-i\omega' V(u_0)} \frac{\omega' A}{\omega + \omega' A} \\ &\quad - \frac{1}{2\pi k}\sqrt{\omega'/\omega}\int_0^{1-A} ds\, (1-s)^{i\omega/k} e^{-is\omega'/k}. \end{aligned} \tag{11}$$

If we do the approximation $1 \ll \omega'/k \ll A^{-1}$ and $1 \ll \omega'/\omega \ll A^{-1}$, we arrive at

$$\begin{aligned} \beta^{R,R}_{\omega,\omega'} &\cong \frac{1}{2\pi i\sqrt{\omega\omega'}} \\ &\quad - \frac{1}{2\pi k}\sqrt{\omega'/\omega}\int_0^{1-A} ds (1-s)^{i\omega/k} e^{-is\omega'/k}. \end{aligned} \tag{12}$$

To obtain an explicit expression for the second term on the rhs, we consider the domain

$$D \equiv \{z \in \mathbb{C} | \mathrm{Re} z \in [0, 1-A], \mathrm{Im} z \in [-\epsilon, 0], k/\omega' \ll \epsilon \ll 1\}$$

and, going through the same steps as in [8], we easily obtain that

$$\beta^{R,R}_{\omega,\omega'} \cong \frac{1}{2\pi i \sqrt{\omega\omega'}} e^{-i\omega'/k} \left(\frac{ik}{\omega'}\right)^{i\omega/k} \Gamma\left(1 + i\omega/k\right). \tag{13}$$

Finally, using $|\Gamma\left(1+i\omega/k\right)|^2 = \frac{\pi\omega/k}{\sinh(\pi\omega/k)}$ (see [15]), we get the announced result, for a perfectly reflecting mirror, that

$$\left|\beta^{R,R}_{\omega,\omega'}\right|^2 \cong \frac{1}{2\pi\omega' k}\left(e^{2\pi\omega/k} - 1\right)^{-1}. \tag{14}$$

(3b) Partially reflecting moving mirror. First, we search for the co-moving coordinates (τ, ρ), that is, the coordinates for which the mirror is at rest, τ being the proper time of the mirror, and we take ρ such that its trajectory is given by $\rho = 0$. Introducing the light-like coordinates $(\bar{u}, \bar{v})$, defined as

$$\bar{u} \equiv \tau - \rho, \quad \bar{v} \equiv \tau + \rho, \tag{15}$$

we will calculate the mirror's trajectory in the coordinates $(\bar{u}, \bar{v})$. Along this trajectory, the length element obeys the identity [16]

$$d\tau^2 = d\bar{u}^2 = d\bar{v}^2 = V'(u)du^2 = U'(v)dv^2. \tag{16}$$

Then, an easy calculation yields the relations

$$\bar{u}(u) \equiv \begin{cases} u, & \text{if} \quad u \le 0, \\ \frac{2}{k}(1 - e^{-k\frac{u}{2}}), & \text{if} \quad 0 \le u \le u_0, \\ \bar{u}(u_0) + \sqrt{A}(u - u_0), & \text{if} \quad u \ge u_0, \end{cases} \tag{17}$$

and

$$\bar{v}(v) \equiv \begin{cases} v, & \text{if} \quad v \le 0, \\ \frac{2}{k}(1 - \sqrt{1 - kv}) & \text{if,} \quad 0 \le v \le v_0, \\ \bar{v}(v_0) + A^{-\frac{1}{2}}(v - v_0), & \text{if} \quad v \ge v_0. \end{cases} \tag{18}$$

When the mirror is at rest, scattering is described by the S-matrix (see [3, 17] for more details)

$$S(\omega) = \begin{pmatrix} s(\omega) & r(\omega)e^{-2i\omega L} \\ r(\omega)e^{2i\omega L} & s(\omega) \end{pmatrix}, \tag{19}$$

where $x = L$ is the position of the mirror. This S matrix is taken to be real in the temporal domain, causal, unitary, and the identity at high frequencies [3]. Correspondingly, the 'in' modes in the coordinates $(\bar{u}, \bar{v})$ are (see also [9])

$$\begin{aligned} g^{in}_{\omega,R}(\bar{u},\bar{v}) &= \frac{1}{\sqrt{4\pi|\omega|}} s(\omega) e^{-i\omega\bar{v}} \theta(\bar{u} - \bar{v}) \\ &\quad + \frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega\bar{v}} + r(\omega)e^{-i\omega\bar{u}}\right)\theta(\bar{v} - \bar{u}), \\ g^{in}_{\omega,L}(\bar{u},\bar{v}) &= \frac{1}{\sqrt{4\pi|\omega|}}\left(e^{-i\omega\bar{u}} + r(\omega)e^{-i\omega\bar{v}}\right)\theta(\bar{u} - \bar{v}) \\ &\quad + \frac{1}{\sqrt{4\pi|\omega|}} s(\omega) e^{-i\omega\bar{u}} \theta(\bar{v} - \bar{u}). \end{aligned} \tag{20}$$

Note that the 'in' modes in the coordinates (u, v), namely ϕ^{in}, are defined in the right null past infinity domain $\mathcal{J}_R^-$ by

$$\phi_{\omega,R}^{in} = \frac{1}{\sqrt{4\pi|\omega|}} e^{-i\omega v}, \quad \phi_{\omega,L}^{in} = 0, \tag{21}$$

and in the left null past infinity domain $\mathcal{J}_L^-$ by

$$\phi_{\omega,R}^{in} = 0, \quad \phi_{\omega,L}^{in} = \frac{1}{\sqrt{4\pi|\omega|}} e^{-i\omega u}. \tag{22}$$

From this definition, it is clear that $\bar{g}_{\omega,k}^{in}(u,v) \equiv g_{\omega,k}^{in}(\bar{u}(u),\bar{v}(v))$, with $k = R, L$, are *not* such modes. However, the modes $\bar{g}_{\omega,k}^{in}$ constitute in fact an orthonormal basis of the space of solutions to our problem. Consequently, if we use the fact that $\bar{g}_{-\omega,k}^{in} = \bar{g}_{\omega,k}^{in*}$, we obtain the following relation

$$\phi_{\omega,k}^{in} = \int_{\mathbb{R}} d\omega' \chi(\omega')(\bar{g}_{\omega',k}^{in}; \phi_{\omega,k}^{in})\bar{g}_{\omega',k}^{in}, \tag{23}$$

with $\chi(\omega')$ the sign function. To be remarked is the fact that Eq. (23) is to be interpreted as follows:

$$\phi_{\omega,k}^{in} = \lim_{\lambda\to\infty} \int_{\mathbb{R}} d\omega' \chi(\omega')(\bar{g}_{\omega',k}^{in}; \phi_{\omega,k}^{in})\bar{g}_{\omega',k}^{in} F_\lambda(\omega'), \tag{24}$$

where $F_\lambda(\omega')$ is a frequency cut-off, for instance $\frac{\lambda^2}{\lambda^2+(\omega')^2}$.

To calculate explicitly the 'in' modes, we choose the coefficients:

$$r(w) = \frac{-i\alpha}{\omega + i\alpha}, \qquad s(w) = \frac{\omega}{\omega + i\alpha}, \tag{25}$$

with $\alpha \geq 0$. In this case, on the rhs of the mirror we obtain

$$\begin{aligned} \phi_{\omega,R}^{in}(u,v) &= \frac{1}{\sqrt{4\pi|\omega|}} e^{-i\omega v} + \phi_{\omega,R}^{refl}(u), \\ \phi_{\omega,L}^{in}(u,v) &= \phi_{\omega,L}^{trans}(u), \end{aligned} \tag{26}$$

where

$$\phi_{\omega,R}^{refl}(u) = \begin{cases} \frac{1}{\sqrt{4\pi|\omega|}} \frac{-i\alpha}{\omega+i\alpha} e^{-i\omega V(u)}, \quad u \leq 0, \\ \\ \frac{1}{\sqrt{4\pi|\omega|}} \frac{-i\alpha}{\omega+i\alpha} e^{-\alpha\bar{u}(u)} - \frac{2\alpha}{k\sqrt{4\pi|\omega|}} e^{-i\frac{\omega}{k}} \int_0^{\frac{k}{2}\bar{u}(u)} \\ ds\, e^{\frac{i\omega}{k}\left(s+1-\frac{k}{2}\bar{u}(u)\right)^2} e^{-\frac{2\alpha s}{k}}, \quad 0 \leq u \leq u_0, \\ \\ \frac{1}{\sqrt{4\pi|\omega|}} \frac{-i\alpha}{\omega+i\alpha} e^{-\alpha\bar{u}(u)} - \frac{1}{\sqrt{4\pi|\omega|}} \frac{i\alpha}{\sqrt{A}\omega+i\alpha} \\ \times \left[e^{-i\omega V(u)} - e^{-i\omega V(u_0)} e^{-\alpha(\bar{u}(u)-\bar{u}(u_0))}\right] \\ -\frac{2\alpha}{k\sqrt{4\pi|\omega|}} e^{-i\frac{\omega}{k}} e^{-\alpha(\bar{u}(u)-\bar{u}(u_0))} \int_0^{\frac{k}{2}\bar{u}(u_0)} \\ ds\, e^{\frac{i\omega}{k}\left(s+1-\frac{k}{2}\bar{u}(u_0)\right)^2} e^{-\frac{2\alpha s}{k}}, \quad u \geq u_0, \end{cases} \tag{27}$$

and

$$\phi^{trans}_{\omega,L}(u)=\begin{cases}\frac{1}{\sqrt{4\pi|\omega|}}\frac{\omega}{\omega+i\alpha}e^{-i\omega V(u)}, \quad u\leq 0\\ \\ \frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega u}+\frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\alpha}{\omega+i\alpha}e^{-\alpha\bar{u}(u)}-\frac{2\alpha}{k\sqrt{4\pi|\omega|}}\\ \int_0^{\frac{k}{2}\bar{u}(u)}ds(s+1-\frac{k}{2}\bar{u}(u))^{2i\omega/k}e^{-\frac{2\alpha s}{k}}, \, 0\leq u\leq u_0\\ \\ \frac{1}{\sqrt{4\pi|\omega|}}\frac{-i\alpha}{\omega+i\alpha}e^{-\alpha\bar{u}(u)}+\frac{1}{\sqrt{4\pi|\omega|}}\frac{e^{-i\omega u_0}}{\omega+i\alpha\sqrt{A}}\\ \left[\omega e^{-i\frac{\omega}{\sqrt{A}}(\bar{u}(u)-\bar{u}(u_0))}+i\alpha\sqrt{A}e^{-\alpha(\bar{u}(u)-\bar{u}(u_0))}\right]\\ -\frac{2\alpha}{k\sqrt{4\pi|\omega|}}e^{-\alpha(\bar{u}(u)-\bar{u}(u_0))}\int_0^{\frac{k}{2}\bar{u}(u_0)}ds\\ (s+1-\frac{k}{2}\bar{u}(u_0))^{2i\omega/k}e^{-\frac{2\alpha s}{k}}, \quad u\geq u_0.\end{cases} \tag{28}$$

Note that (as already advanced) in the case of perfect reflection, that is, when $\alpha\to\infty$, we get

$$\phi^{refl}_{\omega,R}(u)\to-\frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega V(u)}, \quad \phi^{trans}_{\omega,L}(u)\to 0, \tag{29}$$

and when the mirror is transparent, i.e. $\alpha\to 0$, we obtain

$$\phi^{refl}_{\omega,R}(u)\to 0, \quad \phi^{trans}_{\omega,L}(u)\to\frac{1}{\sqrt{4\pi|\omega|}}e^{-i\omega u}. \tag{30}$$

We are interested in the particle production on the rhs of the mirror, for this reason we must now obtain, for $\omega,\omega'>0$,

$$\beta^{R,R}_{\omega,\omega'}=(\phi^{out\,*}_{\omega,R};\phi^{refl}_{\omega',R})^*, \quad \beta^{R,L}_{\omega,\omega'}=(\phi^{out\,*}_{\omega,R};\phi^{trans}_{\omega',L})^*. \tag{31}$$

We start by calculating $\beta^{R,R}_{\omega,\omega'}$, with the result

$$\begin{aligned}\beta^{R,R}_{\omega,\omega'} \;\cong\; & \frac{1}{2\pi\sqrt{\omega\omega'}}\frac{\alpha}{\omega'+i\alpha}\left[1-\frac{\alpha}{k}\int_A^1 dx\, x^{i\omega/k-\frac{1}{2}}e^{-\frac{2\alpha}{k}(1-\sqrt{x})}\right]\\ &+\frac{\alpha}{2\pi ki\sqrt{\omega\omega'}}e^{-i\omega'/k}\int_A^1 dx\, x^{i\omega/k-\frac{1}{2}}e^{i\frac{\omega'}{k}x}\\ &\times\left[1-\frac{2\alpha}{k}\int_0^{1-\sqrt{x}}ds\, e^{i\frac{\omega'}{k}(s^2+2s\sqrt{x})}e^{-\frac{2\alpha s}{k}}\right].\end{aligned} \tag{32}$$

Now, provided that $\omega'\ll\alpha$, then Eq. (32) turns into Eq. (12). Consequently, we precisely obtain the same behavior as for a perfectly reflecting mirror. However, in the case $\alpha\ll\omega'$, we observe that

$$\beta^{R,R}_{\omega,\omega'}\cong\frac{\alpha}{2\pi ki\sqrt{\omega\omega'}}e^{-i\omega'/k}\left(i\frac{k}{\omega'}\right)^{i\omega/k+\frac{1}{2}}\Gamma\left(\frac{1}{2}+i\omega/k\right), \tag{33}$$

and using the identity $|\Gamma\left(\frac{1}{2}+i\omega/k\right)|^2=\pi/\cosh\left(\pi\omega/k\right)$ (cf. [15]), we conclude that

$$\left|\beta^{R,R}_{\omega,\omega'}\right|^2 \;\cong\; \frac{1}{2\pi k\omega}\left(\frac{\alpha}{\omega'}\right)^2\left(e^{2\pi\omega/k}+1\right)^{-1}. \tag{34}$$

Finally, a simple but rather cumbersome calculation yields the result

$$\left|\beta^{R,L}_{\omega,\omega'}\right|^2 \cong 0, \quad \omega' \ll \alpha, \tag{35}$$

and

$$\left|\beta^{R,L}_{\omega,\omega'}\right|^2 \sim \frac{1}{\omega\omega'}\mathcal{O}\left[\left(\frac{\alpha}{\omega'}\right)^2\right], \quad \alpha \ll \omega'. \tag{36}$$

Note that, in the case $\alpha \ll \omega'$ we indeed obtain the nice feature that the number of created particles in the ω mode, together with the radiated energies, are both finite quantities when $u_0 \to \infty$, in perfect agreement with the conclusions in [7]. More precisely, for a partially transmitting mirror the number of produced particles in the ω mode, namely $\mathcal{N}_\omega$, is approximately

$$\mathcal{N}_\omega \cong \int_0^\infty d\omega' \left|\beta^{R,R}_{\omega,\omega'}\right|^2. \tag{37}$$

In order to calculate this quantity, we split the domain $[0,\infty)$ into two disjoints sets, $[0,k)$ and $[k,\infty)$. In the second domain we can do the approximation (34), and we obtain

$$\int_k^\infty d\omega' \left|\beta^{R,R}_{\omega,\omega'}\right|^2 \cong \frac{1}{2\pi\omega}\left(\frac{\alpha}{k}\right)^2\left(e^{2\pi\omega/k}+1\right)^{-1}. \tag{38}$$

In the other domain, assuming that $k \ll 1$, we have $\omega' \ll 1$ and thus for incident waves of a very low frequency the mirror behaves like a perfect reflector; for this reason we can use the formula (11). Then, a simple calculation yields

$$\int_0^k d\omega' \left|\beta^{R,R}_{\omega,\omega'}\right|^2 \sim \mathcal{O}\left(\frac{k^2}{\omega(\omega^2+k^2)}\right), \tag{39}$$

and, since $k \ll 1$, we conclude that the number of produced particles in the ω mode is approximately

$$\mathcal{N}_\omega \cong \frac{1}{2\pi\omega}\left(\frac{\alpha}{k}\right)^2\left(e^{2\pi\omega/k}+1\right)^{-1}, \tag{40}$$

and the radiated energy $\mathcal{E} \equiv \int_0^\infty d\omega\hbar\omega\mathcal{N}_\omega$ is, with good approximation,

$$\mathcal{E} \cong \frac{\hbar\alpha^2}{4\pi^2 k}\ln 2. \tag{41}$$

This completes the proof of all the statements above.

4. Final comments. It is necessary to remark that there is a crucial difference with the case $\omega' \ll \alpha$, where the number of radiated particles in the ω mode diverges logarithmically with $u_0 \to \infty$. In this situation the physically relevant quantity is the number of created particles in the ω mode per unit time. This dimensionless quantity is finite and its value is given by [7, 8]

$$\lim_{u_0\to\infty}\frac{1}{u_0}\mathcal{N}_\omega = \frac{1}{2\pi}\left(e^{2\pi\omega/k}-1\right)^{-1}. \tag{42}$$

A second point is that we have started an additional calculation for a bidimensional fermionic model with massless particles [19]. We have found that in this situation the reverse change of

statistics happens, namely the Fermi-Dirac statistics for the completely reflecting case turns into the Bose-Einstein statistics for the partially reflecting, physical mirror.

To finish, note again the remarkable fact that the problem we addressed here could be solved *exactly*, thus successfully completing a challenging program initiated by Barton, Calogeracos, and Nicolaevici [9, 10, 11] about ten years ago. As a consequence, the results we have obtained are absolutely solid—they do not hang on a perturbative expansion or approximation of any sort.

The physical reason for this surprising change of statistics may be found in the fact that the form of the spectrum is actually determined *not* through the statistics of the field but rather by the specific trajectory of the mirror and by its interaction with the radiation field. A related, albeit different, example of a phenomenon of this sort occurs in the case of an electric charge following the same trajectory (1). When the radiation field has spin 1, the radiation emitted by the charge obeys Bose-Einstein statistics, but when a scalar charge, and consequently an scalar radiation field, is considered, the emitted radiation will obey the Fermi-Dirac statistics [18].

Finally, we must point out for completeness that another situation where a somehow related feature occurs (but maybe of a different kind) is when measuring the spectrum of a scalar field using a DeWitt detector [13, 14] which follows a uniformly accelerated world-line in Minkowski space-time. In this case, when the dimension of the space-time is even, the Bose-Einstein statistics is obtained, however when this dimension is odd the reverse change of statistics, to the Fermi-Dirac one, takes place [20, 21].

Acknowledgments. This work was supported in part by MEC (Spain), projects MTM2005-07660-C02-01 and FIS2006-02842, and by AGAUR (Generalitat de Catalunya), grant 2007BE-1003 and contract 2005SGR-00790. The investigation is partly based on work done while EE was on leave at the Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755, USA.

References

[1] S.A. Fulling and P.C.W. Davies, Proc. Roy. Soc. Lond. **A348**, 393 (1976).

[2] P.C.W. Davies and S.A. Fulling, Proc. Roy. Soc. Lond. **A356**, 237 (1977).

[3] J. Haro and E. Elizalde, Phys. Rev. Lett. **97**, 130401 (2006); Phys. Rev. **D76**, 065001 (2007); [arXiv:0705.0597].

[4] W.R. Walker, Phys. Rev. **D31**, 767 (1985).

[5] R.D. Carlitz and R.S. Willey, Phys. Rev. **D36**, 2327 (1987).

[6] A. Calogeracos, J. Phys. A: Math. Gen. **35**, 3435 (2002).

[7] N. Nicolaevici, J. Phys. A: Math. Gen. **36**, 7667 (2003).

[8] J. Haro, J. Phys. A: Math. Gen. **38**, L307 (2005).

[9] G. Barton and A. Calogeracos, Ann. Phys. (NY) **238**, 227 (1995).

[10] N. Nicolaevici, Class. Quantum Grav. **18**, 619 (2001); **18**, 2895 (2001).

[11] A. Calogeracos, J. Phys. A: Math. Gen. **35**, 3415 (2002).

[12] D. Buchholz and K. Fredenhagen, Commun. Math. Phys. 84, 1 (1982). R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Second Ed. (Springer, Berlin, 1992).

[13] B.S. DeWitt, Phys. Reports **19**, 295 (1975).

[14] N.D. Birrell and C.P.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, 1982).

[15] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (New York: Dover, 1972).

[16] N. Obadia and R. Parentani, Phys. Rev. **D64**, 044019 (2001).

[17] M.-T. Jaekel and S. Reynaud, J. Phys. **I1**, 1395 (1991).

[18] A.I. Nikishov and V.I. Ritus, J. Exp. Theor. Phys. **81**, 615 (1995).

[19] J. Haro and E. Elizalde, *Black body collapse simulated by vacuum fluctuations with moving semi-transparent mirror* (in preparation, 2007).

[20] S. Takagi, Prog. Theo. Phys. Supp. **88**, 1 (1986).

[21] W.G. Unruh, Phys. Rev. **D34**, 1222 (1986).